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ANALYSIS OF LAMINAR-FLOW HEAT TRANSFER IN THE ENTRANCE REGION OF CIRCULAR TUBES

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Abstract—Non-linear equations of laminar flow of a viscous incompressible fluid in the entrance region of a circular tube have been solved by an exact numerical method to obtain the velocity of the flow in this region. This velocity distribution is used in solving the energy equation numerically to obtain temperature profiles under constant wall temperature and also under constant heat flux at the wall. The local Nusselt number is calculated and the results are compared with those given by other workers.

NOMENCLATURE

a, radius of the tube;
x, axial distance from the entrance section;
r, radial distance;
p, pressure above that of the inlet;
u, axial component of velocity;
ρ, fluid density;
μ, coefficient of viscosity;
ν, kinematic viscosity;
u_0, velocity at the inlet;
t, temperature;
k, thermal conductivity;
t_w, constant wall temperature;
q_w, constant heat flux at the wall;
t_0, temperature at the inlet;
t_m, bulk mean temperature;
v, radial component of velocity.

Non-dimensional quantities

Re, Reynolds number $\frac{u_0 a}{\nu}$;

X, $= \frac{x}{aR}$;

R, $= \frac{r}{a}$;

U, $= \frac{u}{u_0}$;

V, $= \frac{vRe}{u_0} = \frac{va}{\nu}$;

P, $= \frac{p}{(\frac{1}{2}\rho u_0^2)}$;

Pr, Prandtl number, $\frac{uc}{k}$;

θ, $= \frac{t - t_0}{t_w - t_0}$;

T, $= \frac{k(t - t_0)}{aq_w}$;

N_1, local Nusselt number, $= N_2$

$$\frac{2a\left(\frac{\partial t}{\partial r}\right)_{(r=a)}}{(t_w - t_m)}.$$

1. INTRODUCTION

ULRICHSON and Schmitz [1] have obtained numerical solutions for the problem of simultaneous development of velocity and temperature profiles in the case of laminar flow of an incompressible viscous flow in the entrance region of a circular tube. The values of the axial component of velocity were taken from the work of Langhaar [2] and those of the radial components were obtained from the equation of continuity and Langhaar's profiles. Thus, these calculations were a refinement of the work of Kays [3], who also used Langhaar's veloicity profiles to calculate the temperature profiles but, in his calculations, neglected the effect of the radial component of veloicity. The main aim of Ulrichson and Schmitz was to study the effect of this refinement and their calculations have shown a significant difference in the local Nusselt number in the entrance region.

In Langhaar's method the equations of flow, which are of boundary layer type, are first linearized with the help of the velocity at the entrance and the resulting equations are solved analytically. Since the velocity profile downstream differ considerably from those at the inlet, this procedure introduces error in the velocity across those sections which are farther away downstream of the entrance section. A finite-difference technique introduced by Bodia and Osterle [4] has recently been used by several workers. Even in this scheme the momentum equations are linearized at any section $X = X_1$, by means of the velocity at $X = X_1 - \Delta X$. Recently, a refinement of this procedure was proposed by the author [5], where the non-linear equations were solved iteratively and that appears to avoid any error introduced due to linearization.

The purpose of the present study is to estimate the improvement in the results due to the refinement introduced by Ulrichson and Schmitz [1] and to ascertain as to how far their method leads towards an improvement of the approximation to the exact solution of the problem. It is, therefore, necessary to obtain an exact solution of the momentum equation by solving the non-linear equations numerically. In this paper the method given by the author [5] has been modified to include symmetric boundary conditions on the axis of the tube and the energy equation is then solved numerically. A higher order integra-formula has been used to integrate the equation along the radial direction. Most other details remain the same.

In Section 2 the equations and boundary conditions are discussed and in Section 3 the method of solution is given. Some mathematical details are given in appendix. Finally the results are compared with those given by other workers

2. THE EQUATIONS AND BOUNDARY CONDITIONS

The momentum equation, the equation of continuity and the energy equation for the flow in the entrance region are given by,

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial r} = -\frac{1}{\rho}\frac{\mathrm{d}p}{\mathrm{d}x} + \frac{\nu}{r}\frac{\partial}{\partial r}\left(r\frac{\partial u}{\partial r}\right), \quad (1)$$

$$\frac{\partial u}{\partial x} + \frac{1}{r}\frac{\partial}{\partial r}(rv) = 0$$

$$\text{or } 2\pi\int_0^a ru\,\mathrm{d}r = u_0\pi a^2, \quad (2)$$

$$\rho c\left(u\frac{\partial t}{\partial x} + v\frac{\partial t}{\partial t}\right) = \frac{k}{r}\frac{\partial}{\partial r}\left(r\frac{\partial t}{\partial r}\right). \quad (3)$$

The boundary conditions are

$$\begin{aligned} &u = u_0, v = 0, p = 0 \text{ for } x = 0, \quad 0 \leqq r < a; \\ &u = v = 0 \text{ for } x \geqslant 0, r = a; \\ &\frac{\partial u}{\partial r} = 0 \text{ for } x \geqslant 0, \quad r = 0. \end{aligned} \quad (4)$$

Depending upon the prescribed constant wall temperature or prescribed constant heat flux at the wall, two types of boundary conditions are considered for the temperature equation (3). In both these cases the boundary conditions at $r = 0$ and at $x = 0$ are

$$\begin{aligned} &t = t_0 \text{ for } x = 0, \qquad 0 \leqq r < a \\ &\frac{\partial t}{\partial r} = 0 \text{ for } x \geqslant 0, \qquad r = 0. \end{aligned} \quad (5)$$

For constant wall temperature we have

$$t = t_w, \text{ for } x \geqslant 0, \qquad r = a \quad (6)$$

and we shall call this problem 1. Similarly for constant heat flux at the wall we have

$$\frac{\partial t}{\partial r} = \frac{q_w}{k} \text{ for } x \geqslant 0, r = a \tag{7}$$

and this will be called problem 2.

Introducing non-dimensional quantities X, R, U, V, P as well as θ for problem 1 and T for problem 2, we get the following equations:

$$U\frac{\partial U}{\partial X} + V\frac{\partial U}{\partial R} = -\frac{1}{2}\frac{\mathrm{d}P}{\mathrm{d}X} + \frac{\partial^2 U}{\partial R^2} + \frac{1}{R}\frac{\partial U}{\partial R}, \tag{8}$$

$$\frac{\partial V}{\partial X} + \frac{1}{R}\frac{\partial}{\partial R}(RV) = 0 \text{ or } \int_0^1 RU\,\mathrm{d}R = \frac{1}{2}, \tag{9}$$

$$U\frac{\partial \theta}{\partial X} + V\frac{\partial \theta}{\partial R} = \frac{1}{Pr}\left(\frac{\partial^2 \theta}{\partial R^2} + \frac{1}{R}\frac{\partial \theta}{\partial R}\right). \tag{10}$$

For problem 2 we obtain an equation similar to (10) in which θ is replaced by T. The boundary conditions are given by,

for $X \geqslant 0$; $U = V = 0$ at $R = 1$

$$\text{and } \frac{\partial U}{\partial R} = V = 0 \text{ at } R = 0, \tag{11}$$

at $\quad X = 0, U = 1, V = 0$ for $0 \leqq R < 1$.

For problem 1 we have,

for $X \geqslant 0, \theta = 1$ at $R = 1$ and

$$\frac{\partial \theta}{\partial R} = 0 \text{ at } R = 0 \tag{12}$$

also at $X = 0, \theta = 0$ for $0 \leqq R < 1$.

For problem 2 we have

for $x \geqslant 0, \frac{\partial T}{\partial R} = 1$ at $R = 1$

$$\text{and } \frac{\partial T}{\partial R} = 0 \text{ at } R = 0, \tag{13}$$

also at $\quad X = 0, T = 0$ for $0 \leqq R < 1$.

We shall solve equations (8–10) under boundary conditions (11) for equations (8) and (9) together with the appropriate boundary conditions, viz (12) or (13), for equation (10).

3 METHOD OF SOLUTION

The derivatives in the X-direction are replaced by forward finite-differences, while the other quantities are replaced by their averages. Let us suppose that the solution $U = U_1$, $P = P_1$, $\theta = \theta_1$ at $X = X_1$ is known and that we wish to find the solution $U = U_2$, $P = P_2$, and $\theta = \theta_2$ at $X = X_2$, where $X_2 > X_1$ and $X_2 - X_1 = l$, say. We now make the following substitutions:

$$U = \frac{U_1 + U_2}{2}, U_1 + U_2 = W\ \frac{\partial U}{\partial X} = \frac{U_2 - U_2}{l},$$

$$\frac{\mathrm{d}P}{\mathrm{d}X} = \frac{P_2 - P_1}{l}, \theta = \frac{\theta_1 + \theta_2}{2},$$

$$\phi = \theta_1 + \theta_2, \frac{\partial \theta}{\partial X} = \frac{\theta_2 - \theta_1}{l},$$

also from (9) and (10)

$$V = -\frac{1}{R}\int_0^R R\frac{\partial U}{\partial X}\mathrm{d}R = -\frac{1}{lR}\int_0^R R\,(U_2 - U_1)\,\mathrm{d}R$$

$$= \frac{1}{lR}\int_0^R R\,(W - 2U_1)\,\mathrm{d}R.$$

Introducing these substitutions in equations (8–10) and denoting derivatives with respect to R by dashes, we get, after some re-arrangement, the following equations:

$$lW'' - W^2 - P_2 + \frac{W'}{R}\int_0^R RW\ \mathrm{d}R$$

$$= -P_1 - \frac{l}{R}W' - 2U_1 W + \frac{2W'}{R}\int_0^R RU_1\,\mathrm{d}R, \tag{14}$$

$$\int_0^1 RW\mathrm{d}R = 1, \tag{15}$$

$$l\frac{\phi''}{Pr} + \frac{l\phi'}{RPr} - lV\phi' - W\phi = -2W\theta_1. \tag{16}$$

In the case of problem 2, i.e. the cast of constant heat flux, we replace θ by T and ϕ by ψ where, $\psi = T_1 + T_2$. The resulting equation is similar to (16) with ϕ replaced by ψ and θ_1 by T_1. The boundary conditions (11) are then replaced by

$$W = 0 \text{ for } R = 1 \text{ and } \frac{\partial W}{\partial R} = 0 \text{ for } R = 0, \quad (17)$$

and the boundary conditions (12) by

$$\phi = 2 \text{ for } R = 1 \text{ and } \frac{\partial \phi}{\partial R} = 0 \text{ for } R = 0, \quad (18)$$

while the conditions (13) by

$$\frac{\partial \psi}{\partial R} = 2 \text{ for } R = 1 \quad \text{and} \quad \frac{\partial \psi}{\partial R} = 0 \text{ for } R = 0. \quad (19)$$

We shall first describe an iterative procedure for solving equations (14) and (15), under the boundary conditions (17), to obtain W and P_2. From these, one can easily determine $U_2 = W - U_1$. During the process of computation one also obtains V. Equation (16), being linear is then solved by a straightforward numerical procedure.

Let W_m, $P_{2,m}$ be an approximate solution of equation (14) satisfying (15) and (17). We shall use this solution to linearize equation (14) and obtain W_{m+1}, $P_{2,m+1}$ as a solution satisfying (15), (17) and also the equation

$$lW''_{m+1} - W_{m+1}W_m - P_{2,m+1} + \frac{W'_m}{R}\int_0^R RW_{m+1}\,dR = -P_1 - \frac{l}{R}W'_m - 2U_1W_m + \frac{2W'_m}{R}\int_0^R RU_1\,dR. \quad (20)$$

As discussed by Leigh [6], this method of linearization leads to a convergent iterative process for W and P_2. The process is repeated until

$$|W_{m+1} - W_m| < \varepsilon,$$

where ε depends upon the accuracy desired.

Equations (20), as well as (16), are linear and are solved by the usual finite-difference technique of subdividing the interval $0 \leqslant R \leqslant 1$ in n equal parts of size h ($nh = 1$). The integrals appearing in (20) are also evaluated numerically at each mesh point and the equations (20) and (15) together are replaced by a set of $(n + 1)$ simultaneous linear algebraic equations.

$$\mathbf{AW} = \mathbf{C}$$

where $\mathbf{A}$ is an $(n + 1) \times (n + 1)$ matrix and $\mathbf{W}$ and $\mathbf{C}$ are $(n + 1)$ column vectors. $\mathbf{A}$ and $\mathbf{C}$ are known while $\mathbf{W}$ is given by

$$\mathbf{W}^T = (-P_{2,m+1}, W_{m+1,1}, W_{m+1\,2}, \cdots, W_{m+1,n})$$

where $W_{m+1,j}$ denotes the value of W_{m+1} at $R = h(j - 1)$. Similarly equation (16) and boundary conditions (18) and (19) are replaced by a set of linear algebraic equations. These sets of equations are solved by point. Gauss–Seidel iterative method after rearrangement. Detailed derivation of these equations is given in the appendix.

Finally, when the iterations converge, we get the value of W at $X = X_2$, as well as the values of ϕ and ψ at $X = X_2$. The same process can then be applied to find the solution at $X = X_2 + l$ and so on.

Before proceeding to the next step we also calculate the local Nusselt number N_1 given by

$$Nu = N_1 = \frac{2a(\partial t/\partial r)_{(r=a)}}{(t_w - t_m)}$$

where t_m is the mean temperature.
In case of problem 1 then

$$Nu = N_1 = \frac{2(\partial \theta/\partial r)_{(R=1)}}{(1 - \theta_m)}$$

where $\theta_m = 2\int_0^1 U\theta R\,dR$.

Similarly for problem 2, we have

$$Nu = N_2 = \frac{2}{(T_w - T_m)}$$

with $T_m = 2\int_0^1 UTR\,dR$.

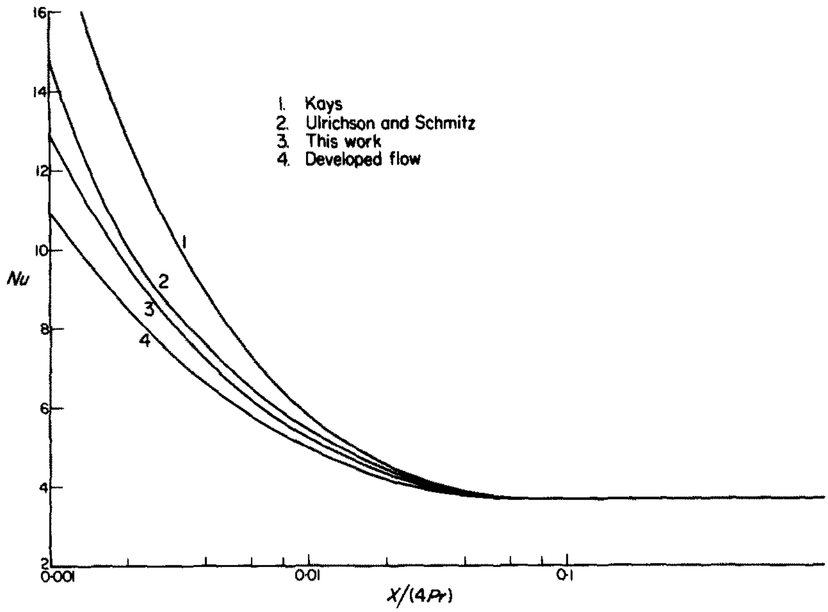

FIG. 1. Local Nusselt number for constant wall temperature and $Pr = 0{\cdot}7$.

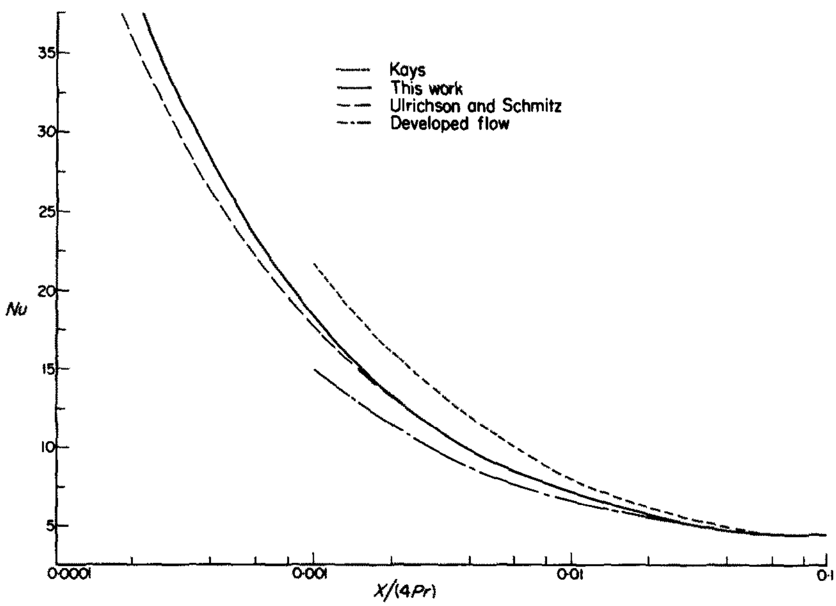

FIG. 2. Local Nusselt number for constant wall flux and $Pr = 0{\cdot}7$.

4. RESULTS

The main aim of this paper is to obtain by a more accurate method, local Nusselt numbers for the heat-transfer problem with constant wall temperature and also for the problem with constant heat flux, and to compare the results with those given by Ulrichson and Schmitz [1]. However, during the process of calculation one obtains without any extra effort and more accurately than by any other method known so far, the velocity profiles, the pressure distribution, an estimate of entrance length, the temperature profiles and various other results.

Near the entrance $X = 0$, the mesh size l was taken very small ($= 0.00005$), since the variation of the velocity in this region is large This was then gradually increased up to 0·0016. Although the method is stable for any mesh ratio $\beta = l/h^2$, h was chosen such that $\beta = l/h^2 < \frac{1}{2}$ for smallest l. To ascertain whether the mesh size was reasonable, the computations were repeated by taking double mesh size in X. The value of ε for convergence was chosen to be 10^{-6} and the number of iterations varied from 8 in the first step to 4 after a few steps. The computations were carried on up to a point $X = 0{\cdot}2145$, where the velocity at two neighbouring points differed by unity in the fourth decimal place. For the calculation of temperature profiles, the Prandtl number Pr was assumed to be 0·7. The computations were done on an IBM 7040 and took little less than an hour.

From an overall heat balance up to any point X, we have

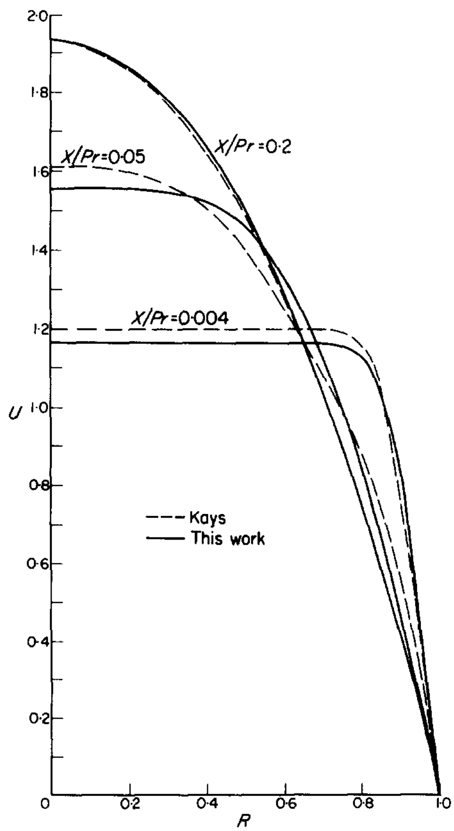

FIG. 3. Generalized velocity profiles for constant wall temperature and $Pr = 0{\cdot}7$.

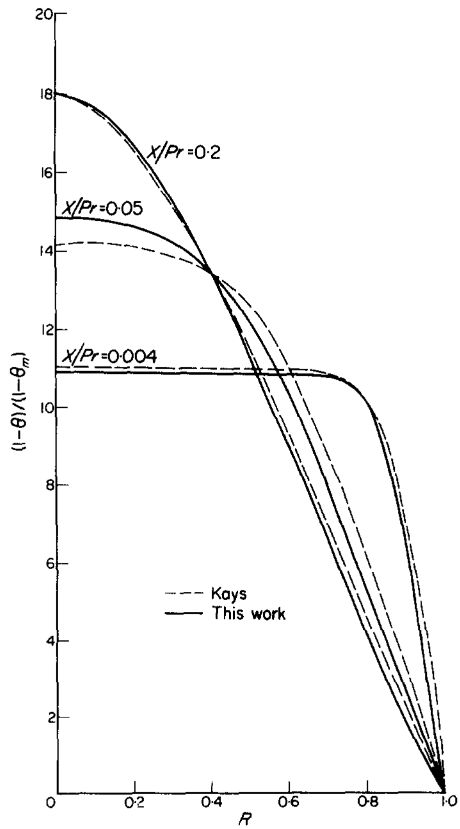

FIG. 4. Generalized temperature profiles for constant wall temperature and $Pr = 0{\cdot}7$.

$$T_m = 2X/Pr$$

which is independent of the velocity distribution and hence can be used as a check on the accuracy of numerical solution. The maximum difference $T_m - 2X/Pr$ was found to be 0·006 at $X = 0{\cdot}2$. The formula used for numerical integration of T_m is given in appendix. The values of local Nusselt numbers together with those obtained by other workers are shown in Figs. 1 and 2. Both the components of velocity, the pressure distribution, the shear-stress at the wall, the values of θ. $T, \theta_m, T_m, N_1, N_2$, etc, were all obtained as out-put. Some velocity and temperature profiles are shown in Figs. 3 and 4.

5. CONCLUSION

Comparison of the results in Figs. 1 and 2 show that so far as local Nusselt number is concerned, the modification introduced by Ulrichson and Schmitz improve the results considerably. In case of constant heat flux the results obtained by Ulrichson and Schmitz are in excellent agreement with those obtained here, for $X/Pr > 0{\cdot}008$. In the earlier stages of flow development the discrepancy is still large. From these comparisons it appears that the effect of radial component of velocity is more significant than those due to linearization of equations of motion except in the immediate neighborhood of the entrance section. However, if the velocity, the pressure and temperature distributions are required more accurately than are given by Langhaar's velocity profiles, the finite-difference technique yields better results [4, 5]. The same procedure as described here can also be used for other problems of this nature.

6. ACKNOWLEDGEMENT

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APPENDIX

Let us divide the interval $0 \leqq R < 1$ into n equal parts (choose n even) and denote W_m at $R = jh$ by $W_{m,j+1}$. From (17) then

$$W_{m,n+1} = W_{m+1,n+1} = 0,$$
$$W'_{m+1,1} = W'_{m,1} = 0.$$

In equation (20) we replace W''_{m+1} and W'_m by

$$W''_{m+1} = (W_{m+1,k+1} - 2W_{m+1,k} + W_{m+1,k-1})/h^2$$

$$\frac{W'_m}{R} = \frac{W_{m,k+1} - W_{m,k-1}}{2h^2(k-1)} = \frac{3\alpha_k}{h^2},$$
$$k = 2, 3, \cdots, n;$$

also $\alpha_1 = 0$. Using Simpson's rule and also the boundary condition at $R = 0$ we can write

$$\int_0^R RU_1\,dR = \int_0^{(k-1)h} RU_1\,dR = \frac{h^2}{3}B_k$$

where

$$B_1 = 0, \quad B_2 = \tfrac{3}{4}(U_{1,1} + U_{1,2}),$$
$$B_3 = \tfrac{3}{5}(-U_{1,1} + 8U_{1,2} + 3U_{1,3})$$

and

$$B_k = B_{k-1} + (k-1)U_{1,k} + 4(k-2)U_{1,k-1} + (k-3)U_{1,k-2}$$
$$k = 4, 5, \cdots, n+1.$$

Define

$$C_k = -P_1 - 3\beta\alpha_k - 2U_{1,k}W_{m,k} + 2\alpha_k B_k \qquad k = 1, 2, \cdots n,$$

and

$$C_{n+1} = 2B_{n+1} \quad \text{and} \quad \beta = l/h^2.$$

By introducing these quantities, equation (20) can be replaced by $\mathbf{AW} = \mathbf{C}$, where

$$C^T = \{C_1, C_2, \cdots, C_{n+1}\}$$

and the matrix $\mathbf{A}$ can easily be written down.

As before we can also write

$$\int_0^R RW \,\mathrm{d}R = \int_0^{(k-1)h} RW \,\mathrm{d}R = \frac{h^2}{3} A_k$$

so that

$$V_k = \frac{(2B_k - A_k)}{3h(k-1)\beta} \quad \text{for} \quad k = 1, 2, \cdots n.$$

In the energy equation (16) we replace ϕ'' and ϕ' by

$$\phi'' = (\phi_{k+1} - 2\phi_k + \phi_{k-1})/h^2 \quad \text{and} \quad \phi' = (\phi_{k+1} - \phi_{k-1})/2h.$$

Using these finite differences and the boundary conditions $\phi_{n+1} = 2$ and $\phi'_0 = 0$ from (18) we can replace equation (16) by a set of simultaneous algebraic equations.

For problem 2 the boundary conditions (19) are replaced by $\psi_2 = \psi_0$ and $\psi_{n-1} - 4\psi_n + 3\psi_{n+1} = 4h$.

For finding the mean values θ_m and T_m we use the same integration formula as was used for $\int_0^R RU_1 \,\mathrm{d}R$, because the conditions at $R = 0$ are the same.

Résumé—Les équations non-linéaires de l'écoulement laminaire d'un fluide visqueux incompressible dans la région d'entrée d'un tube circulaire ont été résolues par une méthode numérique exacte pour obtenir la vitesse de l'écoulement dans cette région. Cette distribution de vitesse est employée pour résoudre numériquement l'équation de l'énergie afin d'obtenir les profils de température avec des températures pariétales constantes et aussi avec un flux de chaleur constant à la paroi. Le nombre de Nusselt local est calculé et les résultats sont comparés avec ceux donnés par d'autres chercheurs.

Zusammenfassung—Die nichtlinearen Gleichungen für die laminare Strömung einer zähen, inkompressiblen Flüssigkeit im Anglauf eines Kreisrohres wurden mit einer genauen numerischen Methode gelöst, um die Strömungsgeschwindigkeit in diesem Bereich zu ermitteln. Diese Geschwindigkeitsverteilung wurde bei der Lösung der Energiegleichung verwendet, um die Temperaturprofile bei konstanten Wandtemperaturen, wie auch bei konstanter Wärmestromdichte an der Wand zu erhalten. Die daraus berechnete örtliche Nusselt-Zahl wird mit den Ergebnissen anderer Autoren verglichen.

Аннотация—Для определения полей скорости на входном участке круглой трубы с помощью точного численного метода решены нелинейные уравнения ламинарного течения вязкой несжимаемой жидкости. Эти распределения скорости используются при численном решении уравнения энергии для получения профилей температур в случаях постоянной температуры стенки и постоянном тепловом потоке. Рассчитано локальное значение критерия Нуссельта и проведено сравнение с результатами других исследователей.